Int. J. Heat Mass Transfer. Vol. 17, pp. 869-873. Pergamon Press 1974. Printed in Great Britain

# ANALYSIS OF INSTABILITIES IN ROUND PIPES AND INFINITE CHANNELS BY STOCHASTIC METHODS

P. HRYCAK
Mechanical Engineering Department, Newark College of Engineering, Newark, N.J., U.S.A.

(*Received* 19 *July* 1973)

**Abstract**—The author proposes in the paper an application of the methods based on the principles of irreversible thermodynamics to describe the conditions under which laminar viscous flow becomes unstable thermodynamically. It is shown that this "point of instability" may be calculated for the round pipe and infinite channel flow according to the so-called Meissner's entropy principle.

The actual undelayed transition may be approximately calculated by a trivial modification of the original Meissner's entropy principle according to the theorem of the mean, and taking account of the fact that during an idealized transition (at a constant mean velocity and without any temperature change) the actual kinetic energy of the fluid based on the steady-state velocity distribution must be considered separately for each regime.

## NOMENCLATURE

$a$, coefficient in expression for $f_{turb}$;
$A$, area;
$b$, exponent in expression for $f_{turb}$;
$d$, half width of channel;
$D$, pipe diameter;
$f$, flow friction coefficient for viscous flow;
$G$, coefficient in expression for $f_{lam}$;
$KE$, $= \int_A \frac{1}{2}u^3\, dA/UA$, based on actual velocity distribution during each fraction of cycle, 1/7 velocity profile during turbulent period;
$L$, significant length;
$p$, pressure;
$Re$, Reynolds number, $UL\rho/\mu$;
$s$, entropy;
$u$, local velocity in the $x$ direction;
$U$, average velocity in the $x$ direction;
$x$, distance from pipe entrance;
$y$, dimension perpendicular to flow direction;
$\gamma$, fraction of cycle time flow stays turbulent;
$\mu$, dynamic viscosity;
$\rho$, density;
$\tau$, shear stress; $\tau_{turb} = a(Re_L)^{-b}\frac{1}{2}\rho U^2 y/L$;
$\langle\ \rangle$, average value.

### Subscripts

$D, d$, significant length on which $Re$ is based;
$cr$, signifies the critical Reynolds number;
$i$, signifies the point where intermittent flow starts;
lam, turb, signifies the character of the flow regime;
$o$, conditions at the origin.

## INTRODUCTION

So FAR, attempts to treat transition in round pipes and infinite channels mathematically have not been particularly successful. Of some interest, therefore, are the applications of the so-called "Meissner's entropy principle" [1] to the problem of transition [2–4] that seem to show some promise in that respect [5].

Experimental observations (e.g. by Sackmann and his collaborators [6, 7] and by Rotta [8]) have established that the originally laminar pipe flow, as the Reynolds number is increased, starts to fluctuate between periods of alternating laminar and turbulent flow. This "intermittent flow" starts with the intermittency factor of $\gamma = 0^+$ at $Re_i$, called here "the point of instability", and ends at $\gamma = 1$, when the flow became completely turbulent.

The actual flow behavior before and after transition can be re-constituted, for example, from the integration of (ideally) fully laminar and fully turbulent periods which quickly follow each other [6–8]. The transition is then observed when the time-averaged properties (e.g. the pressure drop) exhibit a characteristic change of slope, at the instant that the lower critical Reynolds number, $Re_{cr}$, has been reached as (ideally) the end result of infinitesimal changes in $Re$, as shown, e.g. in Fig. 1.

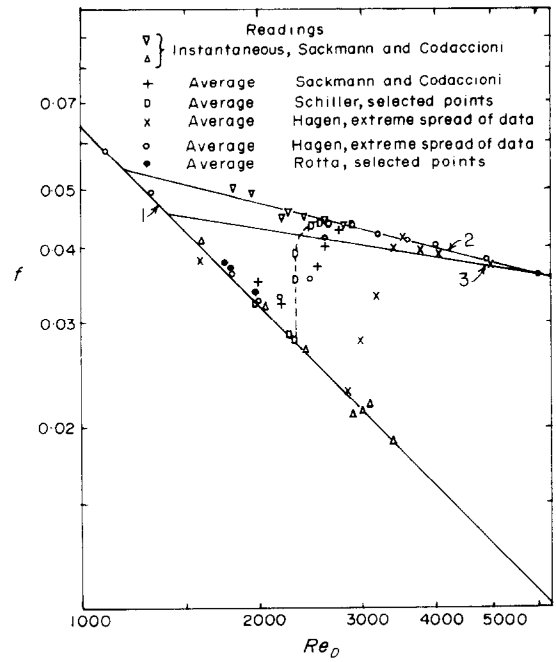


FIG. 1. Instantaneous and average values of pipe friction factor $f$ in transition region by selected authors; Schiller's gap between disturbing plate and pipe entrance 0·6–1·2 mm, and $x/D \to \infty$; Curve 1 = Hagen–Poiseuille formula for $f_{lam}$, curve 2 = Blasius formula for $f_{turb}$, and curve 3 = Meissner and Schubert formula for $f_{turb}$ adapted for low $Re_D$ values.

## STATEMENT OF THE PROBLEM

According to the Meissner's entropy principle [1], at the time of transition (considered to be an isothermal process) when the quasi-steady, fully developed flow has been reached,

$$(\Delta s)_{lam} = (\Delta s)_{turb} \tag{1}$$

which for an isothermal transition in a round pipe or an infinite channel may be expressed by the usual methods of reversible thermodynamics, for an incompressible fluid, as

$$\Delta s = \Delta p/(T\rho) \tag{2}$$

which, however, for viscous flow could also be written as

$$\Delta s = f\rho U^2(x-x_0)/(2T\rho L). \tag{3}$$

For $f_{lam} = G/Re_L$, and $f_{turb} = a(Re_L)^{-b}$ combining equations (1) and (2) results in

$$Re_L = (G/a)^{1/(1-b)} \tag{4}$$

where, for a round pipe, $L = D$, $G = 64$, $a = 0{\cdot}3164$, $b = \frac{1}{4}$ (based on the Blasius formula for viscous friction [5]), and $a = 0{\cdot}153$, and $b = 1/6$ (according to a formula developed by Meissner and Schubert [1] (M & S formula here)), equation (4) yields $Re_D = 1200$ (when Blasius formula is used) and $Re_D = 1400$ (when M & S formula is used). The M & S formula was considered by the authors as a more appropriate one for the relatively low range of Reynolds numbers where non-delayed transition occurs (Fig. 1).

Following the experimental work of Sackmann *et al.* [6, 7], it appears that for instantaneous value of $f_{turb}$ the Blasius relation is quite satisfactory down to $Re_D < 2000$, well below the range of its steady-state validity. However, due to strongly empirical nature of both formulas for $f_{turb}$, they may be used both to at least mark off the range where at first the point of instability $Re_i$, and then the critical Reynolds number, $Re_{cr}$, is likely to occur during a non-delayed transition.

It is interesting to note that extrapolation of Rotta's data to $\gamma = 0$ produces a Reynolds number range which is not inconsistent with the above results if the point of instability is accepted to be in the range $1200 < Re_{D,i} < 1400$. Moreover, Whan and Rothfuss [9] have independently observed at $Re_D = 1200$ the start of a small, but definite, upward trend from the theoretical value of $f_{lam}$. Results that appeared before somewhat suspicious (cf. discussion of some of Knodel's conclusions [10] and of the Reichardt's result [11] $Re_{D,cr} = 1500$ by Schiller [12]) acquire an entirely new significance if the Meissner's entropy principle in its original form is considered as an indicator of the point of instability, $Re_i$, and not of $Re_{cr}$.

It is well known that in round pipes transition actually occurs [5, 12] for $2000 < Re_{D,cr} < 2300$. Assuming the basic physical soundness of the Meissner's entropy principle, and using some results of irreversible thermodynamics, the following modifications are suggested below, to produce $Re_{D,cr}$ more in line with the experimental results.

## THE THEOREM OF MINIMUM ENTROPY PRODUCTION AND TRANSITION

The entropy production in the steady state is according to some authorities [13], with $F$ being a so-called "dissipation function",

$$P[S] = F/T \tag{5}$$

whereas from the results of Millikan one can conclude that for the Navier–Stokes equations for incompressible, steady state flow the laminar viscous dissipation represents a minimum for the cases of pipe, infinite channel, and Couette flow [14]. His results can be summarized as

$$\delta \int_A E\,\mathrm{d}A = -\delta(x-x_0) \int_A \left\{ \tfrac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 \mu + u\frac{\partial p}{\partial x} \right\} \mathrm{d}A = 0. \tag{6}$$

Letting $u' = \partial u/\partial y$, and using the Eulerian equation

$$\frac{\partial}{\partial y}\left(\frac{\partial E}{\partial u'}\right) - \frac{\partial E}{\partial u} = 0 \qquad (7)$$

there results

$$\mu \frac{\partial}{\partial y}\left(y^i \frac{\partial u}{\partial y}\right) - y^i \partial p/\partial x = 0 \qquad (8)$$

which equation for $i = 1$ represents the Hagen–Poiseuille flow, and for $i = 0$ the flow in an infinite channel. For turbulent flow, the dissipation function $\frac{1}{2}F_{\rm lam} = \mu\int_A \frac{1}{2}(\partial u/\partial y)^2\,{\rm d}A$ should be replaced by $F_{\rm turb} = \int_A \tau(\partial u/\partial y)\,{\rm d}A$, and a result analogous to equation (8) will be obtained if $\tau$ is kept constant during the variational process, as suggested by Corrsin [15]. The above discussion illustrates the point that, in the steady state, the entropy production represents an extremum, which physically can be only a minimum.

During the process of transition, even if the mean velocity $U$ is kept constant [8], fluctuations in flow regimes occur, where, due to a rearrangement of velocity profiles, there occur kinetic energy changes like, e.g. $(\Delta KE)_{\rm lam} = 2{\cdot}0\langle KE\rangle$, $(\Delta KE)_{\rm turb} = 1{\cdot}05\langle KE\rangle$, with $\langle KE\rangle = \frac{1}{2}U^2$ being the nominal kinetic energy based on the average velocity [16]. It is obvious that this kind of velocity rearrangements leads to an "excess entropy production"; one can proceed formally, according to the theory of the mean, letting $P(S)_{\rm lam} = P(S)_{\rm turb}$ replace equation (1), and

$$P[S]' = P[S] + (\partial P[S]/\partial\langle KE\rangle)\Delta KE \qquad (9)$$

be the true-kinetic energy change modified expression for entropy production of each regime on transition: equation (4) may be replaced by the formula

$$Re_{L,cr} = (G\phi/a)^{1/(1-b)} \qquad (10)$$

(which still must be considered entirely semi-empirical) with $\phi = [1 + (\Delta KE)_{\rm lam}/\langle KE\rangle]/[1 + (\Delta KE)_{\rm turb}/\langle KE\rangle]$. For a round pipe, $\phi = 1{\cdot}46$ and equation (10) yields $Re_{D,cr} = 2000$ (based on the Blasius formula) and $Re_{D,cr} = 2200$ (based on the M & S formula [1]), which compares favorably with the experimentally determined range of critical Reynolds numbers for non-delayed transition in round pipes [5, 12].

For an infinite channel of width $2d$, $G = 12$, $a = 0{\cdot}0790$ (according to Blasius) and 0·0382 (based on the M & S formula [1]), $(\Delta KE)_{\rm lam} = 1{\cdot}54\langle KE\rangle$ and $(\Delta KE)_{\rm turb} = 1{\cdot}045\langle KE\rangle$ [16], so that $\phi = 1{\cdot}24$, and equation (4) yields $800 < Re_{2d,i} < 1000$, while equation (10) results in $1000 < Re_{2d,cr} < 1300$. These findings may be compared, e.g. with the results of Patel and Head [17], who got $Re_{2d,cr} = 1300$, and independently from M & S formula [1], $a = 0{\cdot}0376$, and $b = 1/6$, whereas Monin and Yaglom [18] quote $Re_{2d,cr} = 1000$, based on the older results of Davies and White.

Some additional discussion involving equation (10) and its applications to Couette flow and to calculation of transition involving non-Newtonian mechanics may be found elsewhere [19].

## DISCUSSION OF THE RESULTS

As is shown in Fig. 1, for a sudden transition in a round pipe the amount of disturbance is an important factor. Thus, Schiller [12] obtained experimentally a virtually complete transition at $Re_D = 2300$ under condition of what he called "maximum disturbance" (unter grösster Störung), by means of a flat plate that could be fixed at various distances from the entrance to the test pipe in the direction perpendicular to flow. As distance between the pipe and the plate increased, the transition curve became less steep (cf. Ref. [12]). Some determined efforts have been made to include the magnitude of disturbance in the quantitative handling of transition in pipes (e.g. discussion by Serrin [20]) but the results must still be considered only preliminary.

It appears, therefore, that the semi-empirical method of handling transition by means of equation (10) has some merits in a simplified description of transition as a single "swing" from one regime to another under conditions of continuous entropy production corresponding roughly to the intermittency factor $\gamma = \frac{1}{2}$ [6]. Also, the range of calculated values $2000 < Re_{D,cr} < 2200$ comes reasonably close to those observed experimentally for non-delayed transition $2000 < Re_{D,cr} < 2300$ [5, 12].

The flow fluctuations in the transition regime seem to be what is known as "relaxation oscillations" in non-linear mechanics [21]. Thus, it appears that the normal mode analysis as described, e.g. in [22], could hardly be expected to produce physically satisfactory results in systems exhibiting a high degree of non-linearity. As an example, the results of Lin [22], applied to an infinite channel, produce $Re_{2d,i} = 7080$, about one order of magnitude higher than that obtained by equation (4), and clearly out of line with the physical reality represented by the results of Patel and Head [17] and Davies and White [18].

At any rate, there appears now evidence, even going back to the original results by Hagen (cf., e.g. Fig. 4 of [22]) that $Re_{D,i}$ calculated with the help of Meissner's entropy principle has physical significance, as corresponding to $f_{\rm lam}$ where the first systematic deviations from the Hagen–Poiseuille law could be observed [9–11], while the so-called normal mode analysis gives very satisfactory results in a variety of other important geometries [22].

## CONCLUSIONS

It has been shown that, at least in some geometries where the normal mode analysis does not seem to work, the analysis by stochastic methods has its legitimate place in the mechanics of transition.

It must be stressed, however, that the method discussed in this paper has nothing in common with the so-called delayed transition (with the highest known $Re_{D,cr} = 5 \times 10^5$) which implies a very considerable deviation from the thermodynamic equilibrium [18] but is a phenomenon which is somewhat similar to supersaturation or undercooling [1, 23].

One practical consequence of some additional insight into mechanics of transition are possibilities of calculation of heat transfer in transition region, as it appears in most heat-transfer applications that the occurrence of transition takes place in about the same fashion as it does in the isothermal flow, in the Reynolds number range $2000 < Re_D < 2300$. Moreover, even a cursory examination of current literature [24] shows negligible differences between isothermal and non-isothermal mean velocity distributions and shear stresses.

The results of this paper bear some similarity to those obtained recently by Jones and Launder [25], but the present calculation of transition is closer to the physical reality, and is much simpler mathematically.

Finally, the main conclusions of this paper may be stated as follows: (a) the transition is not isentropic, as has been originally recognized by Meissner and Schubert themselves [1], (b) the theoretically correct theorem of minimum entropy production in the steady state for each regime *ipso facto* requires that there be an "excess entropy production" during transition, and (c) the results of this investigation must be considered by necessity strictly semi-empirical, as long as there is no satisfactory theory concerning turbulent flow in round pipes and infinite channels, and a theoretical explanation concerning the Blasius and the M & S expressions for $f_{turb}$ is not available. In view of the essentially non-deterministic character of such flows according to the present state of the art, such a theory is certainly not going to be available in the near future, if at all.

## REFERENCES


1. W. Meissner and G. U. Schubert, Kritische Reynoldssche Zahl für Rohrströmung und Entropieprinzip, *Ann. Phys.* 6th Ser **3**, 163 (1948).
2. E. Maurer, Die kritische Reynoldssche Zahl für Kreisröhre nach dem Entropieprinzip, *Z. Phys.* **126**, 522 (1949).
3. E. Maurer, Eine neue theoretische und experimentelle Überprüfung des Meissnerschen Entropieprinzips in der Hydrodynamik an Hand der Couette-Strömung, *Z. Angew. Phys.* **4**, 466 (1952).
4. O. Lauer, Laminare und turbulente stationäre Strömung von Luft und Kohlensäure in einem schwach konischen Rohr in der Nähe der kritischen Reynoldsschen Zahl, *Z. Angew. Phys.* **5**, 81 (1953).
5. W. Kaufmann, *Fluid Mechanics*. McGraw-Hill, New York (1963).
6. L. A. Sackmann and F. Codaccioni, Sur les changements de régime dans les canalisations, *C.R. Hébd. Séanc. Acad. Sci., Paris* **224**, 1326 (1947).
7. L. A. Sackmann and E. Pérès, Sur les changements de régime dans les canalisations. Enregistrement continu des charactéristiques, *C.R. Hebd. Séanc. Acad. Sci., Paris* **240**, 2376 (1955).
8. J. Rotta, Experimenteller Beitrag zur Entstehung turbulenter Strömung im Rohr, *Ing.-Arch.* **24**, 258 (1956).
9. G. A. Whan and R. R. Rothfuss, Characteristics of transition flow between parallel plates, *A.I.Ch.E. Jl* **5**, 204 (1959).
10. A. Knodel, Über die Gasströmung in Rohren und den Luftwiderstand von Kugeln, *Ann. Phys.* 4th Ser. **80**, 533 (1926).
11. H. Reichardt, Elektrische Potentiale bei laminarer und turbulenter Strömung, Inauguraldissertation, Göttingen (1928).
12. L. Schiller, *Handbuch der Experimentalphysik*, Teil 4, s. 126. Akad. Leipzig (1932).
13. P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability and Fluctuations*. Wiley-Interscience, London (1971).
14. C. B. Millikan, On the steady motion of viscous, incompressible fluids, with particular reference to a variational principle, *Phil. Mag.* 7th Ser. **7**, 641 (1929).
15. S. Corrsin, Turbulent channel flow from a variational principle, *J. Aeronaut. Sci.* **20**, 357 (1953).
16. J. K. Vennard, *Handbook of Fluid Mechanics*, edited by V. L. Streeter, Part 3. McGraw-Hill, New York (1961).
17. V. C. Patel and M. R. Head, Some observations on skin friction and velocity profile in fully developed pipe and channel flows, *J. Fluid Mech.* **38**, 181 (1969).
18. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*. MIT Press, Cambridge, Mass. (1971).
19. P. Hrycak and M. J. Levy, Critical Reynolds number motivated interaction between test set-up and the environment, *Proceedings of the Institute of Environmental Sciences*, p. 397 (1973).
20. J. Serrin, Mathematical principles of classical fluid mechanics, *Handbuch der Physik*, Teil *VIII/1*, s. 254. Springer, Berlin (1959).
21. J. J. Stoker, *Nonlinear Vibrations*. Interscience, New York (1950).
22. H. Schlichting, Entstehung der Turbulenz, *Handbuch der Physik*, Teil *VIII/1*, s.355, 382. Springer, Berlin (1959).
23. R. C. Binder, *Fluid Mechanics*, 2nd Edn, p. 89. Prentice-Hall, New York (1949).
24. A. K. Kudva and A. Sesonske, Structure of turbulent velocity and temperature fields in ethylene glycol pipe flow at low Reynolds number, *Int. J. Heat Mass Transfer* **15**, 127 (1972).
25. W. P. Jones and B. E. Launder, The calculation of low-Reynolds number phenomena with a two-equation model of turbulence, *Int. J. Heat Mass Transfer* **16**, 1119 (1973).

## ANALYSE DES INSTABILITES DANS DES TUBES CIRCULAIRES ET DES CANAUX PAR DES METHODES STOCHASTIQUES

**Résumé**—L'auteur propose une application des méthodes basées sur les principes de la thermodynamique des phénomènes irréversibles pour décrire les conditions sous lesquelles un écoulement laminaire devient instable thermodynamiquement. On montre que le "point d'instabilité" peut être calculé pour un écoulement dans un tube circulaire dans un canal infini, suivant le principe d'entropie dit de Meissner.

La transition instantanée peut être calculée approximativement par une modification banale du principe d'entropie dû à Meissner, en utilisant le théorème de la moyenne et en tenant compte du fait que, durant une transition idéalisée (à vitesse moyenne constante et sans changement de température), l'énergie cinétique effective du fluide, basée sur la distribution permanente des vitesses, peut être considérée séparément pour chaque régime.

## ANALYSE VON INSTABILITÄTEN IN RUNDEN ROHREN UND UNENDLICHEN KANÄLEN MITTELS STOCHASTISCHER METHODEN

**Zusammenfassung**—Der Autor schlägt in dem Aufsatz vor, Methoden der irreversiblen Thermodynamik zur Beschreibung der Bedingungen zu verwenden, unter denen eine zähe, laminare Strömung thermodynamisch instabil wird. Es wird gezeigt, daß der "Instabilitätspunkt" für die Strömungen in runden Rohren und unendlichen Kanälen entsprechend dem sogenannten Meissnerschen Entropieprinzip berechnet werden kann.

Der momentane, unverzögerte Umschlagspunkt kann näherungsweise berechnet werden durch einfache Modifikation des Meissnerschen Entropieprinzips entsprechend dem Theorem des Mittelwertes sowie unter Berücksichtigung der Tatsache, daß während eines idealisierten Umschlages (bei konstanter Durchschnittsgeschwindigkeit und konstanter Temperatur) die momentane kinetische Energie des Fluids für jeden Vorgang bezogen auf die stationäre Geschwindigkeits verteilung betrachtet werden muß.

## АНАЛИЗ НЕУСТОЙЧИВОСТЕЙ С ПОМОЩЬЮ СТОХАСТИЧЕСКИХ МЕТОДОВ В КРУГЛЫХ ТРУБАХ И НЕОГРАНИЧЕННЫХ КАНАЛАХ

**Аннотация** — В статье для описания условий, при которых ламинарное вязкое течение становится термодинамически неустойчивым, предлагается использовать методы, основанные на принципах термодинамики необратимых процессов. Показано, что на основе так называемого принципа энтропии Мейсснера можно рассчитать «точку неустойчивости» для течения в круглой трубе и неограниченном канале.

Можно приближенно рассчитать момент возникновения неустойчивости с помощью модификации исходного принципа энтропии Мейсснера согласно теореме о среднем. С учетом того факта, что имеется идеализированный переход (при постоянной средней скорости и без изменения температуры) истинную кинетическую энергию жидкости, основанную на стационарном распределении скорости, необходимо рассматривать для каждого режима в отдельности.